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# Lecture 7 - QCD

References: Donoghue, Golowich, Holstein
Cheng and Li

## QCD Confines

Coupling constants are not constant. One can use the Renormalization Group to resum logarithms into a redefinition of a "running coupling" according to an RG equation $\mu \frac{de}{d\mu} = \beta(e)$, where $e$ is the running coupling (e.g the gauge coupling in QED), $\beta(e)$ is the "$\beta$-function" that determines the evolution, and $\mu$ is the RG scale (chosen to be a typical energy scale of the experiment to minimize the logarithms.)

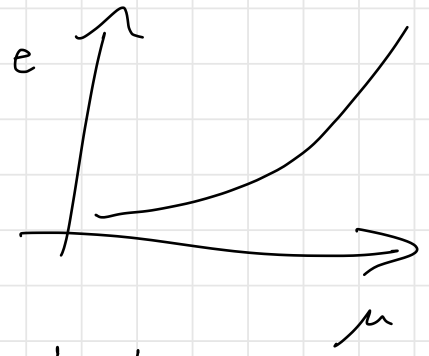
In QED,  $\Rightarrow \beta(e) = \frac{e^3}{12\pi^2} + \mathcal{O}(e^5)$

We can solve this 1-loop RG for $e(\mu)$:

$$\frac{1}{e^3}\,de = \frac{1}{12\pi^2}\frac{1}{\mu}\,d\mu$$

$$\Rightarrow \frac{1}{e^2(\mu)} = \frac{1}{e^2(\mu_0)} - \frac{1}{12\pi^2}\log\left(\frac{\mu}{\mu_0}\right)$$

$$\Rightarrow e^2(\mu) = \frac{e^2}{1 - \frac{e^2}{12\pi^2}\log\left(\frac{\mu}{\mu_0}\right)} \Rightarrow$$




$\Rightarrow$ QED has a "Landau Pole" at high energies.

For QCD with $n_f$ quarks, the renormalized Lagrangian includes

$$\mathcal{I} = \mu^{\frac{4-d}{2}} g_R Z_1 A_\mu^a \psi_i \bar{\gamma}^\mu T^a_{ij}\psi_j = \mu^{\frac{4-d}{2}} g_R \frac{Z_1}{Z_2\sqrt{Z_3}} A_\mu^{a(0)} \bar{\psi}_i^{(0)}\gamma^\mu T^a_{ij}\psi_j^{(0)}$$

bare

$Z_2$ ($Z_3$) is field-strength renorm for $\psi$ ($A$). $(Z = 1+\delta)$

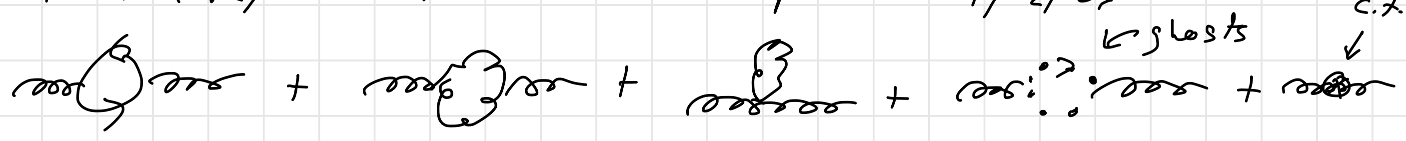
So we identify $g_0 = g\,\dfrac{Z_1}{Z_2\sqrt{Z_3}}\,\mu^{\frac{4-d}{2}}$.

Using $\dfrac{dg_0}{d\mu} = 0 \Rightarrow \mu\dfrac{dg}{d\mu} = \left[-\dfrac{\varepsilon}{2} - \mu\dfrac{d}{d\mu}\left(\delta_1 - \delta_2 - \dfrac{1}{2}\delta_3\right)\right]$

Plugging in the leading order $\mu\dfrac{dg}{d\mu} \Rightarrow$

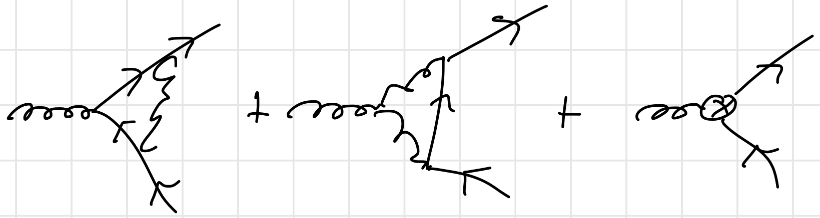
$$\beta(g) = -\frac{\varepsilon}{2}g + \frac{\varepsilon}{2}g^2\frac{\partial}{\partial g}\left(\delta_1 - \delta_2 - \frac{1}{2}\delta_3\right)$$

Therefore, we need to compute $\delta_1, \delta_2, \delta_3$ ← see Schwartz ch. 26




$$\Rightarrow \delta_3 = \frac{1}{\varepsilon}\left(\frac{g^2}{16\pi^2}\right)\left[\frac{10}{3}C_A - \frac{8}{3}n_f T_F\right] \quad \left(\begin{matrix}\text{dim reg} \\ \xi = 1 \text{ gauge}\end{matrix}\right)$$

$$\Rightarrow \delta_2 = \frac{1}{\epsilon}\left(\frac{g^2}{16\pi^2}\right)\left[-2C_F\right]$$



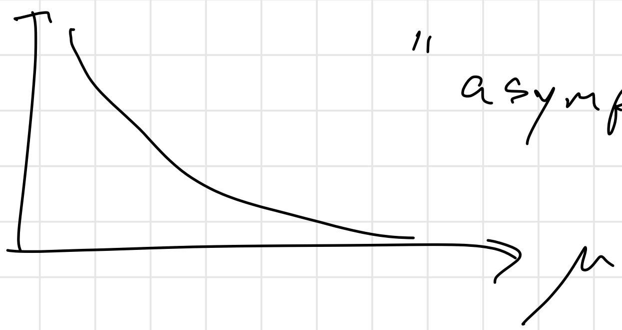
$$\Rightarrow \delta_1 = \frac{1}{\epsilon}\left(\frac{g^2}{16\pi^2}\right)\left[-2C_F - 2C_A\right]$$

Putting it all together and taking $C_A = 3$ and $T_F = \frac{1}{2}$ for QCD, we have (in $d<4$)

$$\beta(g) = -\frac{g^3}{16\pi^2}\left[11 - \frac{2}{3} n_f\right]$$

$\Rightarrow$ If $n_f < 17 \Rightarrow \beta(g) < 0$

$\Rightarrow$ $g$



"asymptotic freedom"

$\Rightarrow$ QCD becomes non-perturbative at low energies. Using $\alpha_s = \frac{g^2}{4\pi} = 0.12$ @ $\mu = m_Z$

$\Rightarrow g \sim 4\pi$ @ $\mu \equiv \Lambda_{QCD} \approx 100$ MeV.

So we see that we can not describe QCD at low energies in terms of quarks and gluons $\Rightarrow$ hadrons are asymptotic states of QCD.

The implication is that the color force "confines" and the asymptotic states are color singlets. Note that since QCD becomes weakly coupled at high energies, we expect that we can describe the quarks and gluons inside a hadron as nearly free particles. We will revisit this.

Two ways to make a color ($SU(3)$) singlet: contract with

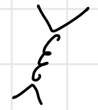
- $\delta_i^{\,j}$ : quark - anti-quark bound state $\Rightarrow$ mesons
- $\varepsilon^{ijk}$ : 3 quark (or 3 anti-quark) bound state $\Rightarrow$ baryons

(See Schwartz 26.2)

Also, need to have the color force be attractive. For quark - anti-quark, use $3 \otimes \bar{3} = 8 \oplus 1$. Color singlet has the form

$$|1\rangle = \frac{1}{\sqrt{3}}\left(|r\bar{r}\rangle + |b\bar{b}\rangle + |g\bar{g}\rangle\right)$$

red
green
blue

Can compute tree-level potential 

$$\Rightarrow \quad V(r) = -\frac{4}{3}\frac{g^2}{4\pi r} \quad \text{(singlet)}$$

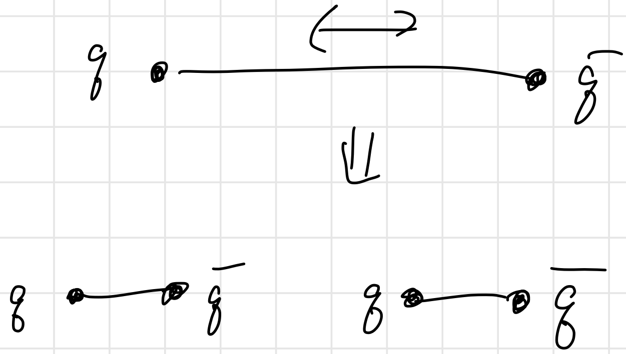
$$V(r) = \frac{1}{6}\frac{g^2}{4\pi r} \quad \text{(octet)}$$

$\Rightarrow$ plausability argument for confined mesons.

Qualitatively, we say the strong force connects quarks by color strings. The energy in these strings grows linearly with the seperation:

$$V(r) \simeq \sigma r \quad \text{w/ } \sigma = \text{string tension}$$

Eventually there is enough energy stored in the string to pop a virtural quark-anti-quark pair out of the vacuum:




* Look up the "Schwinger model": 2-D QED, which exhibits confinment.

$\Rightarrow$ one meson becomes two

$\Rightarrow$ the quarks are always confined

# Light mesons $m \lesssim \Lambda_{QCD}$

mesons are bosons

Now, let us study the mesons that exist in nature.

Note that they are charictorized by their mass, spin, and flavor of constituant quarks.

spin-0
parity odd

There are 9 relatively light pseudoscalar mesons:

$$\underline{\eta'} \qquad 958 \text{ MeV}$$

$$\underline{\eta} \qquad 548 \text{ MeV}$$

strangeness $\longrightarrow$ $\underline{K^-}$ $\underline{\bar{K}^0}$ $\underline{K^0}$ $\underline{K^+}$ 498 MeV

$$\underline{\pi^-} \quad \underline{\pi^0} \quad \underline{\pi^+} \qquad 140 \text{ MeV}$$

9 somewhat hevier vector mesons (spin 1, parity odd)

$$\underline{\varphi^0} \qquad 1020 \text{ MeV}$$

strangeness $\rightarrow$ $\underline{K^{*-}}$ $\underline{\bar{K}^{*0}}$ $\underline{K^{*0}}$ $\underline{K^{*+}}$ 892 MeV

$\underline{\rho^-}$ $\quad$ $\underline{\omega^0}$ $\underline{\rho^0}$ $\quad$ $\underline{\rho^+}$ $\quad$ 781 MeV, 770 MeV

Interpret these light mesons as being built of up, down, strange quarks. 7.7

Note mass pattern of vector mesons, each level is about 120 MeV hevier than the previous one. Each level involves one more strange/anti-strange quark. They are 9 bound states of

$$(\bar{u}, \bar{d}, \bar{s}) \times (u, d, s)$$

The masses of the pseudoscalar mesons are more subtle to understand.

Note, there is an approximate global symmetry (suggested by Heisenberg in 1932 to explain properties of nuclear energy levels from forces mediated by meson exchanges) called "isospin". When acting on only $u + d$, it is the $SU(2)$ group where $u + d$ transform under the fundamental representation: $\begin{pmatrix} u \\ d \end{pmatrix} \to e^{i\vec{\alpha}\cdot\vec{\sigma}/2} \begin{pmatrix} u \\ d \end{pmatrix}$

Can organize $\bar{u}u$, $\bar{u}d$, $\bar{d}u$, $\bar{d}d$ into combinations of $\frac{1}{2}\otimes\frac{1}{2} = 1\oplus 0$ reps

isospin 1 has 3 degenerate states: 3 $\pi$'s + 3 $\rho$s

isospin 0 has 1 state: $\eta$ + $\omega$

Can also treat strange quark as massless

$\Rightarrow$ generalize to $SU(3)$

$$\bar{3}\otimes 3 = 8\oplus 1 \quad \Rightarrow \quad 8 \text{ is "eightfold way"}$$

## Heavy mesons $m \gtrsim \Lambda_{QCD}$

These are mesons with at least one heavy quark: c or b. (Note, no mesons with top quarks since $\Gamma_{top} = 1.3\,GeV > \Lambda_{QCD}$ so it decays before it would "hadronize")

pseudoscalars

strange $\rightarrow$ $D_s^+$    1968 MeV

$D^0$   $D^+$    1869 MeV

vectors

$D_s^{*+}$    2112 MeV

$D^{*0}$   $D^{*+}$    2010 MeV

are $c\bar{u}$, $c\bar{d}$, $c\bar{s}$ bound states.

pseudoscalar

| | | |
|---|---|---|
| | $B_s^0$ | 5367 MeV |
| | $B^-$ $B^0$ | 5279 MeV |
| vector | $B_s^{*0}$ | 5415 MeV |
| | $B^{*-}$ $B^{*0}$ | 5325 MeV |

are $b\bar{u}$, $b\bar{d}$, $b\bar{s}$ bound states.

* With meson systems, we have multiple EFT techniques we can use to model their phenomena. For pions + kaons, we can identify them as Goldstone bosons ⇒ "chiral $\mathcal{L}$" For B mesons (and to some extent D mesons) we can use "Heavy Quark EFT". We also have the non-perturbative numerical tool called "lattice QCD."

Baryons

Baryons are fermions that carry a global charge called baryon number. The proton is a baryon (uud) as is the neutron (udd)

In principle, $p \to e^+ \pi^0$ or $p \to \nu K^+$ (consistent w/ kinematics and electric charge) but violates baryon number. Experimentally,

$$\tau_{p\to e\pi} > 8\times 10^{33}\,\text{yrs} \quad + \quad \tau_{p\to\nu K} > 7\times 10^{32}\,\text{yrs}$$

On the other hand, the neutron can decay

$n \to p e^+ \nu$ via the weak interactions

without violating baryon number

There are 8 spin 1/2 states (baryon octet) and 10 spin 3/2 states (baryon decuplet)

## EFT for light mesons

We know the up, down, and strange quarks are all "small": $m_{u,d,s} < \Lambda_{QCD}$. We can derive a useful EFT description for some of the mesons built out of them by relying on the global symmetries of QCD w/ $m \to 0$. Specifically there is an $SU(n_f)_L \times SU(n_f)_R$ symmetry that rotates the left/right handed fermions in flavor space.

Then we assume that the strong force at low energies spontaneously breaks this global symmetry via a non-zero vacuum expectation value of a quark bilinear: $\langle \bar{\psi}\psi \rangle \sim \Lambda^3_{QCD}$

Since $\bar{\psi}\psi = \psi_L \psi_R + h.c.$, this breaks

$$SU(n_f)_L \times SU(n_f)_R \rightarrow SU(n_f)_V \quad (V = \text{vector})$$

Our goal is to build the EFT for the resulting Goldstone bosons. We can do this using the exponential parametrization $U(x)$, by assuming it transforms as $U \rightarrow L U R^\dagger$ where

$L \in SU(n_f)_L$ and $R \in SU(n_f)_R$

The types of terms we can write are

$$\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger),\ \mathrm{Tr}(\partial_\mu U \partial_\nu U^\dagger) \cdot \mathrm{Tr}(\partial^\mu U \partial^\nu U^\dagger),$$
$$\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger)\, \mathrm{Tr}(\partial_\nu U D \partial^\nu U^\dagger),\ \text{etc.}$$

Note that all terms must involve derivatives since $\mathrm{Tr}(U^\dagger U) = \mathrm{Tr}(\mathbb{1}) = n_f$. The most general $\mathcal{L}$ is

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 + \dots = \frac{F^2}{4}\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger) + l_1 \left[\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger)\right]^2$$
$$+ l_2\, \mathrm{Tr}(\partial_\mu U \partial_\nu U^\dagger)\, \mathrm{Tr}(\partial^\mu U \partial^\nu U^\dagger) + \dots$$

$[F] = 1$ (in 4D), and this is an EFT

where $\mathcal{L}_2 \Rightarrow \mathcal{O}(E^2/F^2)$, $\mathcal{L}_4 \Rightarrow \mathcal{O}(E^4/F^4)$, etc.

## Explicit Symmetry Breaking

We can understand how to incorporate explicit symmetry breaking into the theory. For simplicity, let's specialize to $n_f = 2$. We can start with the linear realization $\Sigma_1' = \sigma + i\,\vec{\tau}\cdot\vec{\pi}$ where (after symmetry breaking) $\sigma$ is the "radial mode", $\vec{\pi}$ are the Goldstones and we have fermion fields $\psi_L$ and $\psi_R$.

The Lagrangian is:

$$\mathcal{L} = \frac{1}{4}\mathrm{Tr}\left(\partial_\mu \Sigma_1' \partial^\mu \Sigma_1'^\dagger\right) + \frac{\mu^2}{4}\mathrm{Tr}\left(\Sigma_1'^\dagger \Sigma_1'\right)$$

$$- \frac{\lambda}{16}\left[\mathrm{Tr}\,\Sigma_1'^\dagger \Sigma_1'\right]^2 + \bar{\psi}_L i\partial\!\!\!/ \psi_L + \bar{\psi}_R i\partial\!\!\!/ \psi_R$$

$$- g\left(\bar{\psi}_L \Sigma_1' \psi_R + \bar{\psi}_R \Sigma_1'^\dagger \psi_L\right)$$

The model is invariant under $SU(2)_L \times SU(2)_R$ where $\psi_L \to L\psi_L$, $\psi_R \to R\psi_R$, $\Sigma_1' \to L\Sigma_1' R^\dagger$

Going to the symmetry breaking vacuum, $\sigma = v + \tilde{\sigma}$ w/ $v = \sqrt{\frac{\mu^2}{\lambda}}$

we have

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \tilde{\sigma}\, \partial^\mu \tilde{\sigma} - 2\mu^2 \tilde{\sigma}^2\right) + \frac{1}{2}\partial_\mu \vec{\pi}\cdot\partial^\mu \vec{\pi}$$

$$- \lambda v \tilde{\sigma}\left(\tilde{\sigma}^2 + \vec{\pi}^2\right) - \frac{\lambda}{4}\left(\tilde{\sigma}^2 + \vec{\pi}^2\right)^2$$

$$+ \bar{\psi}\left(i\partial\!\!\!/ - gv\right)\psi - g\bar{\psi}\left(\tilde{\sigma} - i\vec{\tau}\cdot\vec{\pi}\gamma^5\right)\psi$$

Now rewrite this into non-linear form using

$$\Sigma = \sigma + i\vec{\tau}\cdot\vec{\pi} = (v + S)\, U \quad \text{w/} \quad U = \exp\left(i\vec{\tau}\cdot\vec{\pi}'/v\right)$$

where $\vec{\pi}' = \vec{\pi} + \ldots$

$$\Rightarrow \mathcal{L} = \frac{1}{2}\left[(\partial_\mu S)^2 - 2\mu^2 S^2\right] + \frac{(v+S)^2}{4}\, \mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right)$$

$$- \lambda v S^3 - \frac{\lambda}{4} S^4 + \bar{\psi} i\partial\!\!\!/ \psi - g(v + S)\left(\bar{\psi}_L U \psi_R + \bar{\psi}_R U^\dagger \psi_L\right)$$

we see that $U \to L U R^\dagger$ as promised.

and we only have derivative couplings for $\vec{\pi}'$

Can set $S = 0$ to get $\mathcal{O}(E^2)$ EFT for $U$ (Can you see why?)

- Now let's add a <u>small symmetry breaking term</u>:

$$\mathcal{L}_{\text{breaking}} = a\sigma = \frac{a}{4}\,\mathrm{Tr}\left(\Sigma + \Sigma^\dagger\right)$$

To first order in $a$, this shifts $v = \sqrt{\frac{\mu^2}{\lambda}} + \frac{a}{2\mu^2}$

so that the Goldstone gets a mass

$$m_\pi^2 = a/v.$$

To see this, we can use the linear rep:

$$\mathcal{L}_{\text{breaking}} = \frac{a}{4}(v+S)\,\mathrm{Tr}(U+U^\dagger)$$

$$= \frac{a}{4}(v+S)\,\mathrm{Tr}\left(2-\left(\frac{\vec{\tau}\cdot\vec{\pi}}{v}\right)^2+\dots\right)$$

$$= a(v+S) - \underbrace{\frac{a}{2v}}_{m_\pi^2/2}\vec{\pi}\cdot\vec{\pi} + \dots$$

The chiral SU(2) is slightly broken but the vector symmetry is intact. Setting $S=0$, we get the EFT Lagrangian to $\mathcal{O}(E^2)$:

$$\mathcal{L}_2 = \frac{v^2}{4}\mathrm{Tr}\left[\partial_\mu U \partial^\mu U^\dagger\right] + \frac{m_\pi^2}{4}v^2\,\mathrm{Tr}(U+U^\dagger)$$

There will also be higher order terms like

$$\left[m_\pi^2\,\mathrm{Tr}(U+U^\dagger)\right]^2,\quad m_\pi^2\,\mathrm{Tr}(U+U^\dagger)\cdot\mathrm{Tr}(\partial_\mu U\partial^\mu U^\dagger), \text{ etc}$$

$\Rightarrow$ when $m_\pi^2$ is small compared to $v$, we have a dual expansion in terms of energy and mass.

We can understand this systematically by letting $m_\pi$ transform as a spurion:

$$m_\pi \to L m_\pi R^\dagger \Rightarrow m_\pi U^\dagger \to L m_\pi U^\dagger L^\dagger$$

$\Rightarrow \mathrm{Tr}(m_\pi U^\dagger]$ is an invariant.

To understand the map between QCD and the Theory of Goldstone bosons (the "Chiral $\mathcal{L}$") we can start with 3 flavor QCD with external source fields:

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \bar{\psi} i \not{D} \psi - \bar{\psi} \gamma_\mu \frac{1+\gamma^5}{2} l^\mu \psi - \bar{\psi} \gamma_\mu \frac{1-\gamma^5}{2} r^\mu \psi$$
$$- \bar{\psi}_L (s + ip) \psi_R - \bar{\psi}_R (s - ip) \psi_L$$

where $l_\mu, r_\mu, s, p$ are $3\times3$ source matrices:

$$l_\mu = l^0_\mu \mathbb{1} + l^a_\mu \lambda^a, \quad r_\mu = r^0_\mu \mathbb{1} + r^a_\mu \lambda^a \quad \text{etc.}$$

and $\lambda^a$ are generators of $SU(3)$, The "Gell-Mann matrices"

This reduces to QCD when $l_\mu = r_\mu = p = 0$ and $s = m$ where $m$ is the $3\times3$ quark mass matrix.

Also note we can include the QED coupling by taking $l_\mu = r_\mu = eQA_\mu$.

Then we can take functional derivatives with respect to these external currents to compute matrix elements, e.g.

$$\langle 0 | \bar{\psi}(x) \psi(x) | 0 \rangle = i \left. \frac{\delta \ln Z}{\delta s^0(x)} \right|_{\substack{l=r=p=0 \\ s=m}}$$

We can make the connection to the theory of Goldstone bosons by starting with the effective action of QCD:

$$e^{iW[l_\mu, r_\mu, s, p]} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A \exp\left(i \int d^4x \, \mathcal{L}_{QCD + \text{Sources}}\right)$$

At low energies, we integrate out the heavy dofs and are left with a theory where only the Goldstones propagate:

$$e^{iW[l_\mu, r_\mu, s, p]} = \int \mathcal{D}U \exp\left(i \int d^4x \, \mathcal{L}_{eff}\{U, l_\mu, r_\mu, s, p\}\right)$$

$\mathcal{L}_{eff}$ has exact local $SU(3)$ invariance if we let the external fields transform as gauge fields

$$\psi_L \to L(x)\psi_L \quad , \quad \psi_R \to R(x)\psi_R$$

$$l_\mu \to L(x)\, l_\mu L^\dagger(x) + i\,\partial_\mu L(x) L^\dagger(x)$$

$$r_\mu \to R(x)\, r_\mu R^\dagger(x) + i\,\partial_\mu R(x) R^\dagger(x)$$

$$(s+ip) \to L(x)\,(s+ip)\,R^\dagger(x)$$

for $L(x) \in SU(3)$ and $R(x) \in SU(3)$

For these invariances to hold, we must introduce a covariant derivative $D_\mu U = \partial_\mu U + i l_\mu U - i U r_\mu$

field strengths $L_{\mu\nu} = \partial_\mu l_\nu - \partial_\nu l_\mu + i[l_\mu, l_\nu]$ 7.17

$$R_{\mu\nu} = \partial_\mu r_\nu - \partial_\nu r_\mu + i[r_\mu, r_\nu]$$

so that
$$U \to L(x)\, U\, R^\dagger(x)$$
$$D_\mu U \to L(x)\, D_\mu U\, R^\dagger(x)$$
$$L_{\mu\nu} \to L(x)\, L_{\mu\nu}\, L^\dagger(x)$$
$$R_{\mu\nu} \to R(x)\, R_{\mu\nu}\, R^\dagger(x)$$

The effective action at $\mathcal{O}(E^2)$ only has 2 terms:
$$\mathcal{L}_2 = \frac{F_\pi^2}{4} \mathrm{Tr}\left(D_\mu U D^\mu U^\dagger\right) + \frac{F_\pi^2}{4} \mathrm{Tr}\left(\chi U^\dagger + U \chi^\dagger\right)$$

where $\chi \equiv 2B_0(s+ip)$ and $B_0$ is a constant with $[B_0] = 1$

For $SU(2)$, taking $l_\mu = r_\mu = p = 0$ and $s = m\mathbb{1}$

we have $m_\pi^2 = (m_u + m_d) B_0$

We can imediately identify
$$\langle 0 | \bar{\psi}_i \psi_j | 0 \rangle = - F_\pi^2 B_0 \delta_{ij}$$

and the left-handed current
$$J^k_{L\mu}(x) = - \frac{\partial \mathcal{L}}{\partial l^\mu_k(x)} = -i \frac{F_\pi^2}{2} \mathrm{Tr}\left(\lambda^k U \partial_\mu U^\dagger\right) \text{ etc.}$$

We see a hint of an "isospin" invariance due to the near degeneracy of $(\pi^{+/-}, \pi^0)$, $(\bar{K}^{+/-}, \bar{K}^0)$, $(p, n)$, etc. This leads us to conjecture that $SU(2)_L \times SU(2)_R \longrightarrow SU(2)_V$ spontaneously by some dynamical mechanism.

We can develop evidence for this picture by systematically studying the predictions of the chiral $\mathcal{L}$. We can use the vacuum expectation value of the fermion scalar bilinear

$$\langle 0 | \bar{\psi} \psi | 0 \rangle = \langle 0 | \bar{\psi}_L \psi_R | 0 \rangle + \langle 0 | \bar{\psi}_R \psi_L | 0 \rangle$$

as a measure of chiral symmetry breaking.

Exact isospin symmetry implies

$$\langle 0 | \bar{u} u | 0 \rangle = \langle 0 | \bar{d} d | 0 \rangle$$

This vacuum expectation value impacts the pion mass. Note that if $m_u = m_d = 0$, the pion would be massless. The part of the QCD $\mathcal{L}$ that explicitly breaks the chiral symmetry is the quark mass term $\mathcal{H}_{mass} = -\mathcal{L}_{mass} = m_u \bar{u} u + m_d \bar{d} d$.

To leading order in this symmetry breaking spurion, the pion mass is generated by the expectation value of this Hamiltionan:

See Cheng + Li; p164

$$m_\pi^2 = \langle \pi | m_u \bar{u} u + m_d \bar{d} d | \pi \rangle \leftarrow$$

This is a Conceqhence of $\langle 0 | A_\mu(0) | \pi(p) \rangle = i F p_\mu$
$\Rightarrow \langle 0 | \partial^\mu A_\mu(0) | \pi(p) \rangle = i F m^2$

We also have

$$m_\pi^2 = (m_u + m_d) B_0 \quad \text{and} \quad \langle 0 | \bar{q} q | 0 \rangle = -\frac{\partial \mathcal{Z}}{\partial s^0} = -F_\pi^2 B_0$$

$$\Rightarrow \quad B_0 = \frac{m_\pi^2}{m_u + m_d} + \dots \neq 0 \Rightarrow \langle 0 | \bar{q} q | 0 \rangle \neq 0$$

Note that the claim is that $m_\pi^2 \sim m_{quark}$. This did not have to be the case, since in general

$$m_\pi^2 = (m_u + m_d) B_0 + (m_u + m_d)^2 C_0 + (m_u - m_d)^2 D_0 + \dots$$

But there is no symmetry reason for $B_0 = 0$ and phenomenologically, $B_0 \neq 0$ appears correct.

Let's extend this to the case of $SU(3)$.

This will allow us to make predictions for the mass relations among the light mesons.

The $SU(3)$ theory has 8 Goldstone bosons

$$\pi^+, \pi^-, \pi^0, K^+, K^-, K^0, \bar{K}^0, \eta$$

They are organized in terms of

$U = \exp\{i\lambda\cdot\varphi/F\}$ w/ $\lambda$ are Gell-Mann matrices

$$\Rightarrow \frac{1}{\sqrt{2}}\sum_{a=1}^{8}{}' \lambda^a\varphi^a = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \overline{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}$$

Expanding $\mathcal{L} \supset \frac{F_\pi^2}{4}\,\mathrm{Tr}\left[mU^\dagger + Um^\dagger\right]$ with $m = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix}$

order $\varphi^2 \Rightarrow$

$$m_\pi^2 = B_0(m_u + m_d) \quad , \quad m_{K^{+/-}}^2 = B_0(m_s + m_u)$$

$$m_{K^0}^2 = B_0(m_s + m_d) \quad , \quad m_\eta^2 = \frac{1}{3}B_0(4m_s + m_u + m_d)$$

Defining $m_K^2 = \frac{1}{2}\left(m_{K^\pm}^2 + m_{K^0}^2\right)$ and $\hat{m} = \frac{1}{2}(m_u + m_d)$

$$\Rightarrow \frac{\hat{m}}{m_s} = \frac{m_\pi^2}{2m_K^2 - m_\pi^2} \simeq \frac{1}{26} \Rightarrow$$ u, d masses are very small $\hat{m} \sim 4\,\mathrm{MeV}$, $m_s \sim 100\,\mathrm{MeV}$

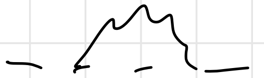
$$m_\eta^2 = \frac{1}{3}\left(4m_K^2 - m_\pi^2\right) \Rightarrow m_\eta = 566\,\mathrm{MeV} \simeq 549\,\mathrm{MeV}$$

↑ observed value

The SU(3) theory is also sensitive to isospin breaking. This was not possible for the SU(2) theory of only pions since to first order $m_{\pi^\pm} - m_{\pi^0}$ is independent of the $\Delta I = 1$ mass difference $m_d - m_u$. However, kaons have a mass splitting at first order in $m_d - m_u$

$$\left(m_{K^0}^2 - m_{K^+}^2\right)_{\text{quark-mass}} = (m_d - m_u) B_0 = \left[\frac{m_d - m_u}{m_s - \hat{m}}\right]\left(m_K^2 - m_\pi^2\right)$$

Note there is also an electromagnetic contribution to the mass splitting. This is quadratically divergent

$$\delta m_\pi^2$$  $$\sim \frac{\alpha}{4\pi} \Lambda_{QCD}^2 \sim (\text{few MeV})^2$$

$\uparrow$ 100 MeV

Measured value is $\sim 5$ MeV.

Note that to leading order, there is no difference in the E&M coupling to pions versus kaons

$$\Rightarrow \left(m_{K^0}^2 - m_{K^+}^2\right)_{em} \simeq m_{\pi^0}^2 - m_{\pi^+}^2$$

$$\Rightarrow \left[\frac{m_d - m_u}{m_s - \hat{m}}\right]\left(m_K^2 - m_\pi^2\right) = \left[\frac{m_d - m_u}{m_d + m_u}\right] m_\pi^2$$

$$= \left(m_{K^0}^2 - m_{K^+}^2\right) - \left(m_{\pi^0}^2 - m_{\pi^+}^2\right)$$

$$\Rightarrow \quad \frac{m_d - m_u}{m_s - \hat{m}} = 0.023 \,, \quad \frac{m_d - m_u}{m_d + m_u} = 0.29$$

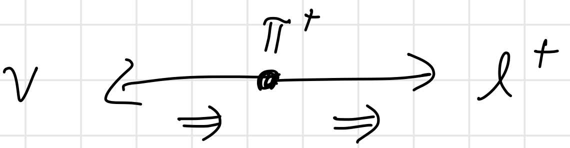
$\Rightarrow m_u / m_d \simeq 0.55 \Rightarrow$ why isospin symmetry holds to a good approximation

## Charged pion decay

The charged pion decays dominantly to $\pi^+ \to \ell^+ \nu_\ell$ through the weak interactions (charged current)

Naively, would expect $Br(e) = Br(\mu)$.

However, recall that the $W$ boson only couples to left handed fermions. Since the pion is spin 0, in the pion rest frame and the $m_\ell = 0$ limit, we have




This violates conservation of angular momentum

$\Rightarrow$ we need to flip the helicity of $\ell^+$, which requires $m_\ell \neq 0 \Rightarrow \dfrac{Br(\mu)}{Br(e)} \sim \dfrac{m_\mu^2}{m_e^2}$

We can compute this using the chiral $\mathcal{L}$ SU(2) theory

Note that we can match the weak Hamiltonian responsible for this decay using

$$\mathcal{H}_W = \frac{G_F}{\sqrt{2}} \underbrace{V_{ud}}_{\text{CKM}} \bar{\psi}_d \gamma_\lambda (1+\gamma^5)\psi_u \Big[\bar{\psi}_{\nu_e}\gamma^\lambda(1+\gamma^5)\psi_e + \bar{\psi}_{\nu_\mu}\gamma^\lambda(1+\gamma^5)\psi_\mu\Big]$$

and $\langle 0 | A^j_\mu(0) | \pi^k(p)\rangle = i F_\pi p_\mu \delta^{ij}$

where $A^j_\mu = \bar{\psi}\gamma_\mu \gamma^5 \frac{\tau^j}{2}\psi$ w/ $\psi = \begin{pmatrix} u \\ d \end{pmatrix}$

Then the matrix element

$$\langle \mu^+ \nu_\mu | \mathcal{H}_W | \pi \rangle = \frac{G_F}{\sqrt{2}} V_{ud} \sqrt{2} F_\pi p_\lambda \bar{u}_\nu \gamma^\lambda (1+\gamma^5) v_\mu$$

$$= -G_F V_{ud} F_\pi m_\mu \bar{u}_\nu (1-\gamma^5) v_\mu$$

$$\Rightarrow \Gamma_{\pi^+\to\mu^+\nu_\mu} = \frac{G_F^2}{4\pi} F_\pi^2 \underbrace{m_\mu^2}_{\text{helicity suppression}} m_\pi |V_{ud}|^2 \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2$$

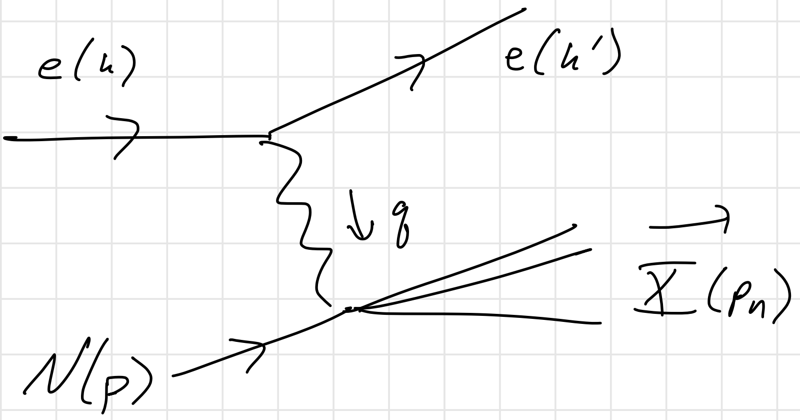
(The neutral $\pi^0$ decay $\pi^0 \to \gamma\gamma$ comes from axial anomaly. Unfortunately, we don't have time to discuss)

# Partons



We have an intuitive picture that hadrons are made of confined quarks and gluons. The strong force becomes strong at low energy = long distance and becomes weak at high energy = short distance. We therefore expect that the quarks inside the hadron behave as essentially free "partons" at short distances, point-like spin 1/2 particles. We can test this idea experimentally by probing the structure of protons and neutrons by scattering leptons off of them. When the lepton scatters off the proton/neutron without destroying the hadron, this is called "elastic" scattering. If the hadron is destroyed, this is "inelastic" scattering. We will see that one prediction of the parton model is that "Bjorken scaling" emerges in the so-called "deep inelastic" regime. Bjorken scaling is the prediction that the momentum fraction exchanged between the lepton and the hadron becomes independent of energy in the deeply inelastic regime.

- Consider the process $e(k) + N(p) \to e(k') + X(p_n)$ where $e$ is an electron, $N$ is a nucleon (either proton or neutron), $X$ is some hadronic final state (could be many hadrons) with total four-momentum $p_n$




Define $M$ = nucleon mass, $q = k - k'$, $\nu = P\cdot q / M$

and we have $p_n^2 = (p+q)^2$

Lab frame: $P_\mu = (M, \vec{0})$, $K_\mu = (E, \vec{k})$, $K'_\mu = (E', \vec{k}')$

$\Rightarrow \nu = E - E'$ is energy loss of the lepton

Neglecting lepton mass:

$$q^2 = (k-k')^2 = -4EE'\sin^2\theta/2 \leq 0$$

w/ $\theta$ is scattering angle of the lepton.

The amplitude is given by

$$\mathcal{M}_n = e^2\, \bar{u}_{h'}(k')\gamma^\mu u_h(k)\, \frac{1}{q^2}\, \langle n| J_\mu^{em}(0)|p,s\rangle$$

where $|p,s\rangle$ is nucleon state w/ spin $s$ and

$|n\rangle \in |X\rangle$ are the hadrons in the final state.

The unpolarized cross section is then

$$d\sigma_n = \frac{1}{|\vec{v}|}\frac{1}{2M}\frac{1}{2E}\frac{d^3k'}{(2\pi)^3 2k_0'}\prod_{i=1}^{n}\frac{1}{4}\sum_{h,h',s}{}' |\mathcal{M}_n|^2 (2\pi)^4\delta^4(p+k-k'-p_n)$$

w/ $p_n = \sum_i p_i$ are the final state hadron momenta.

If we sum all possible hadronic final states (e.g. if we only observe the lepton) we get the "inclusive" cross section

$$\frac{d^2\sigma}{d\Omega\, dE'} = \frac{\alpha^2}{q^4}\left(\frac{E'}{E}\right) l^{\mu\nu} W_{\mu\nu} \qquad \left(\alpha = \frac{e^2}{4\pi}\right)$$

w/ the leptonic tensor

$$l^{\mu\nu} = \frac{1}{2}\,\mathrm{Tr}\left(\not{k}'\gamma^\mu \not{k}\gamma^\nu\right) = 2\left(k^\mu k'^\nu + k'^\mu k^\nu + \frac{q^2}{2}g^{\mu\nu}\right)$$

and the hadronic tensor

$$W_{\mu\nu}(p,q) = \frac{1}{4M} \sum_s{}' \sum_n{}' \int \prod_{i=1}^{n} \left[ \frac{d^3 p_i}{(2\pi)^3 2 p_{i0}} \right]$$

$$\times \langle p, s | J_\mu^{em}(0) | n \rangle \langle n | J_\nu^{em}(0) | p, s \rangle (2\pi)^3 \delta^4(p_n - p - q)$$

$$= \frac{1}{4M} \sum_s{}' \int \frac{d^4x}{2\pi} e^{iq\cdot x} \langle p, s | J_\mu^{em}(x) J_\nu^{em}(0) | p, s \rangle$$

Note that em current conservation implies

$$q^\mu \langle p, s | J_\mu^{em} | n \rangle = 0$$

$$\Rightarrow q^\mu W_{\mu\nu} = q^\nu W_{\mu\nu} = 0$$

Therefore, the most general form $W_{\mu\nu}(p,q)$ can take is

$$W_{\mu\nu}(p,q) = \left[ -W_1 \left( g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) + \frac{W_2}{M^2} \left( p_\mu - \frac{p\cdot q}{q^2} q_\mu \right) \left( p_\nu - \frac{p\cdot q}{q^2} q_\nu \right) \right],$$

where $W_1$ + $W_2$ are Lorentz invariant "structure functions" of $q^2$ and $\nu = p\cdot q / M$

They charictorize the response of the nucleon to electromagnetic probes.

$$\Rightarrow \frac{d^2\sigma}{d\Omega dE'} = \frac{\alpha^2}{4E^2 \sin^4\frac{\theta}{2}} \left( 2W_1 \sin^2\frac{\theta}{2} + W_2 \cos^2\frac{\theta}{2} \right)$$

To explore the meaning of $W_1$ & $W_2$, let us study the case of "elastic" scattering, where $|X\rangle = |N\rangle$.

$$\Rightarrow \langle N(p')|J_\mu^{em}(0)|N(p)\rangle$$

$$= \bar{u}(p')\left[\gamma_\mu F_1(q^2) + \frac{i}{2M}\sigma_{\mu\nu}q^\nu F_2(q^2)\right]u(p)$$

w/ $q = p - p'$ and $F_1$, $F_2$ are Lorentz invariant "form factors". For the proton $F_1^p(0) = 1$ and $F_2^p(0) = 1.79$ measure the total charge and anomalous magnetic moment respectively.

For the neutron, $F_1^n(0) = 0$ and $F_2^n(0) = -1.91$

We can derive the relation between $F_1(0)$ and the total charge as follows. Note that the charge operator acts on the proton state as with momentum $p$ as

$$Q|P(p)\rangle = |P(p)\rangle$$

$$\Rightarrow \langle P(p')|Q|P(p)\rangle = \langle P(p')|P(p)\rangle = 2E(2\pi)^3\delta^3(\vec{p}\,'-\vec{p})$$

We also know $Q = \int d^3x \, J_0^{em}(x)$

$$\Rightarrow \langle P(p')|Q|P(p)\rangle = \int d^3x \langle p'| J_0^{em}(x)|p\rangle$$

$$= \int d^3x \, e^{i(p'-p)\cdot x} \langle p'| J_0^{em}(0)|p\rangle$$

$$= (2\pi)^3 \delta^3(\vec{p}-\vec{p}\,') \bar{u}(p)\gamma_0 u(p) F_1(0)$$

$$= 2E(2\pi)^3 \delta^3(\vec{p}-\vec{p}\,') F_1(0)$$

$$\Rightarrow F_1(0) = 1$$

We can write $W_1$ + $W_2$ in the elastic limit in terms of $F_1$ + $F_2$ w/ $p_n^2 = M^2$:

$$W_1^{el}(q^2,\nu) = \delta(q^2 + 2M\nu)\frac{q^2}{2M} G_M^2(q^2)$$

$$W_2^{el}(q^2,\nu) = \delta(q^2+2M\nu)\frac{2M}{(1-q^2/4M^2)}\left[G_E(q^2) - \frac{q^2}{4M^2}G_M^2(q^2)\right]$$

w/
$$G_E(q^2) = F_1(q^2) + \frac{q^2}{4M^2}F_2(q^2)$$
$$G_M(q^2) = F_1(q^2) + F_2(q^2)$$

are the electric and magnetic form factors.

Then the elastic electron-nucleon cross section is

$$\frac{d\sigma^{el}}{d\Omega} = \frac{\alpha}{4E^2}\left[\frac{\cos^2\frac{\theta}{2}\left(1-\frac{q^2}{4M^2}\right)^{-1}\left(G_E^2 - \frac{q^2}{4M^2}G_M^2\right) - \sin^2\frac{\theta}{2}\frac{q^2}{2M^2}G_M^2}{\left(1+\frac{2E}{M}\sin^2\frac{\theta}{2}\right)\sin^4\frac{\theta}{2}}\right]$$

$\Rightarrow$ Measuring the elastic eN differential cross section yields information about the electric and magnetic form factors.

Experimentally: $G_E(q^2) \simeq \frac{G_M(q^2)}{\aleph} \simeq \frac{1}{(1 - q^2/0.7\,GeV^2)^2}$

w/ $\aleph_p = 2.79$ is magnetic moment of proton.

If the proton were a point particle, $G_M(q^2) = G_E(q^2) = 1$

$\Rightarrow$ $q^2$ dependence implies proton has structure!

Note that elastic cross section drops rapidly since $G_E \sim G_M \sim 1/q^4$ at large $q^2$

Now, we can revisit the inelastic cross section. Suprisingly, the inelastic cross section for large invariant mass has a very moderate $q^2$ dependence. This leads to the idea that there must be point-like constituents inside the nucleon, much like the large angle scattering results seen by Rutherford suggested that the charge in an atom was concentrated in a "point-like" nucleus.

We can make this precise by phrasing it in terms of Bjorkin Scaling.

Define the dimensionless variable $x = \frac{-q^2}{2M\nu}$ which takes the range $0 \leq x \leq 1$

Since $(p+q)^2 = q^2 + 2M\nu + M^2 \geq M^2$

Elastic scattering corresponds to $x = 1$.

Define $y = \frac{\nu}{E} = 1 - \frac{E'}{E}$ is the fraction of initial energy transfered to the hadrons.

Since $0 \leq E' \leq E \Rightarrow 0 \leq y \leq 1$

We can express the general cross section in terms of $x$ and $y$, using

$$dx\, dy = \frac{E'}{E} \frac{d\Omega\, dE'}{2\pi y M}$$

and defining $M W_1(q^2, \nu) = G_1(x, q^2/M^2)$

$$\nu W_2(q^2, \nu) = G_2(x, q^2/M^2)$$

$$\Rightarrow \frac{d^2\sigma}{dx\,dy} = \frac{8\pi\alpha^2}{M E x^2 y^2}\left[xy^2 G_1 + \left(1 - y - \frac{M}{2E}xy\right) G_2\right]$$

Bjorken scaling is the statement that in the large $q^2$ limit with fixed $x$, the $G_i$'s are functions of $x$ only
$$\lim_{\substack{|q^2|\to\infty \\ x \text{ fixed}}} G_i\left(x, \frac{q^2}{M^2}\right) = F_i(x)$$

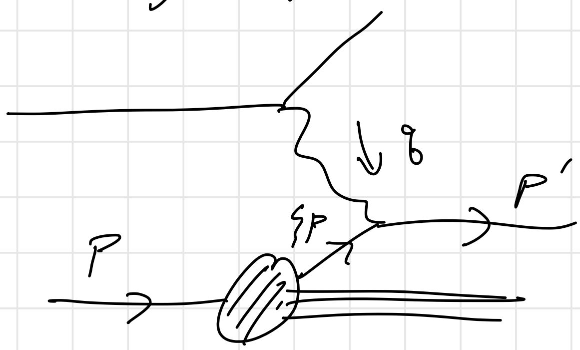
where $F_1(x)$ and $F_2(x)$ are called "scaling functions".
Experimentally, Bjorken scaling seems to hold for $|q^2| \geq 2\ GeV^2$ in $ep$ scattering.

We can derive Bjorken scaling as a consequence of the parton model. We view the inclusive inelastic scattering as the incoherent scattering of point-like constituants of the nucleon. Specifically, we assume

1) we can ignore inter parton interactions during the current-parton interaction
2) the final state interactions associated with fragmenting into hadrons take place on a long enough time scale that we can ignore them

Let a spin 1/2 parton carry a fraction of the original nucleon momentum $p_{parton} = \xi p$ with $0 \leq \xi \leq 1$ (neglect any parton momentum transverse to the nucleon momentum).

The contribution to the $\overline{W}_{\mu\nu}$ hadronic tensor from a single parton is then




$$\overline{W}_{\mu\nu}(\xi) = \frac{1}{4\xi M} \sum_{spin}{}' \frac{d^3p'}{(2\pi)^3\, 2p_0'} (2\pi)^3 \delta^4(p' - \xi p - q)$$

$$\times \langle \xi p, s | J_\mu^{em}(0) | p', s' \rangle \langle p', s' | J_\nu^{em}(0) | p, s \rangle$$

$$= \frac{1}{4\xi M} \sum_{spin} \bar{u}(\xi p) \gamma_\mu u(p') \bar{u}(p') \gamma_\nu u(\xi p)$$

$$\times \delta(p_0' - \xi p_0 - q_0) / 2p_0'$$

Note the $1/\xi$ is due to the relative change in flux going from $p \to \xi p$

We can rewrite the $\delta$-fn as

$$\delta(p_0' - \xi p_0 - q_0)/2p_0' = \Theta(p_0')\,\delta\left[(p_0')^2 - (\xi p_0 + q_0)^2\right]$$
$$= \Theta(p_0')\,\delta\left[(p')^2 - (\xi p + q)^2\right]$$
$$= \Theta(\xi p_0 + q_0)\,\delta(2M\nu\xi + q^2)$$
$$= \Theta(\xi p_0 + q_0)\,\frac{\delta(\xi - x)}{2M\nu}$$

We can write the spin sum as

$$\frac{1}{2}\sum_{spin}{}' \bar{u}(\xi p)\gamma_\mu u(\xi p + q)\,\bar{u}(\xi p + q)\gamma_\nu u(\xi p)$$
$$= \frac{\xi}{2}\,\mathrm{Tr}\left[\not{p}\,\gamma_\mu\,(\xi\not{p} + \not{q})\,\gamma_\nu\right]$$
$$= 2\xi\left(p_\mu(\xi p + q)_\nu + (\xi p + q)_\mu p_\nu - p\cdot(\xi p + q)g_{\mu\nu}\right)$$
$$= 4M^2\xi^2\left(p_\mu p_\nu/M^2\right) - 2M\nu\xi g_{\mu\nu} + \ldots$$

where we have neglected the parton mass.

$$\Rightarrow \bar{K}_{\mu\nu}(\xi) = \delta(\xi - x)\left(\frac{\xi}{\nu}\frac{p_\mu p_\nu}{M^2} - \frac{1}{2M}g_{\mu\nu} + \ldots\right)$$

Next, we can define the number of partons with momenta between $\xi$ and $\xi + d\xi$ as $f(\xi)\,d\xi$

"parton distribution function"

$$\Rightarrow W_{\mu\nu} = \int_0^1 f(\xi)\,\bar{K}_{\mu\nu}(\xi)\,d\xi$$
$$= \frac{x f(x)}{\nu}\frac{p_\mu p_\nu}{M} - \frac{f(x)}{2M}g_{\mu\nu} + \ldots$$

We see that the $\delta$-fn that enforces the on-shell condition leads to the structure function dependence on $x = -q^2/2M\nu$ alone

$$\Rightarrow M W_1 \to F_1(x) = \tfrac{1}{2} f(x)$$

$$\nu W_2 \to F_2(x) = x f(x)$$

so the scaling functions measure the parton distribution functions in this simple approximation.

Note $2x F_1(x) = F_2(x)$ "Callan-Gross relation" is a consequence of assuming the partons were spin $1/2$. If they had been spin-0, we would have had

$$K_{\mu\nu} \sim \langle \xi p | J_\mu | \xi p + q\rangle \langle \xi p + q | J_\nu | \xi p \rangle$$

$$\sim (2\xi p + q)_\mu (2\xi p + q)_\nu$$

$$\Rightarrow W_{\mu\nu} \sim (2xp + q)_\mu (2xp + q)_\nu$$

$$\Rightarrow F_1(x) = 0 \quad \text{for scalar partons}$$

Contracting with polarization vectors

$$\varepsilon_\mu^+ = \frac{1}{\sqrt{2}}(0, 1, i, 0) \qquad \varepsilon_\mu^- = \frac{1}{\sqrt{2}}(0, 1, -i, 0)$$

$$\varepsilon_\mu^L = \frac{1}{\sqrt{-q^2}}(P, 0, 0, E)$$

We can define $W_s = \varepsilon_\mu^{s*} W^{\mu\nu} \varepsilon_\nu^{s'}$

$\Rightarrow 2M W_L \to F_L = \frac{1}{x} F_2 - F_1$

$M W_+ \to F_+ = F_1 - \frac{1}{2} F_3$

$M W_- \to F_- = F_1 + \frac{1}{2} F_3$

($F_3$ defined as coeff of $\varepsilon^{\mu\nu\rho\sigma}$ term)

Shows up for charged current interaction $\nu p \to l n$ scattering

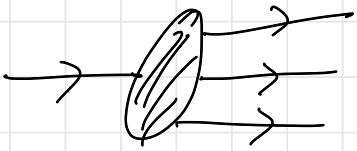
$\Rightarrow F_T = F_+ - F_- = 2F_1$

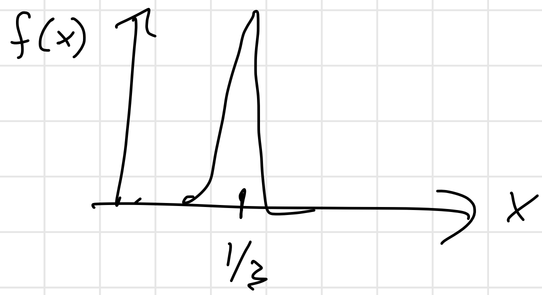
So we can distinguish

$F_S = 0$ for spin-1/2 parton

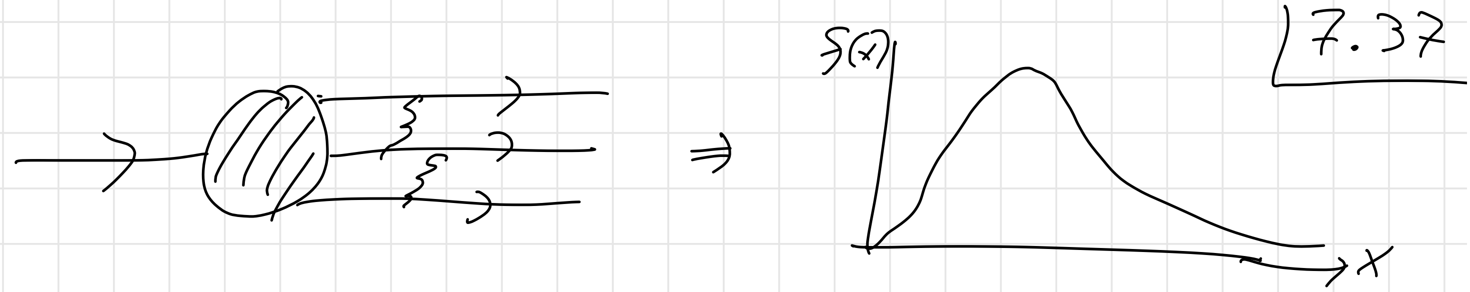
$F_T = 0$ for spin 0 parton

To this order, the parton model would predict 3 valence quarks:





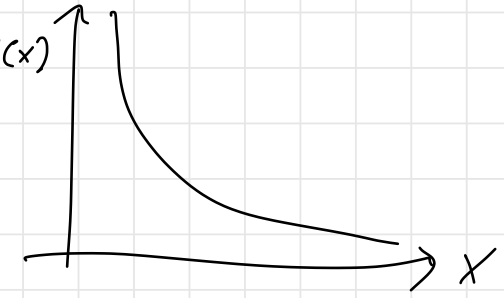

We can refine the model to include corrections due to gluon exchange, which would smear this away from a δ-fn approx:



Finally, there are also $g \to q\bar{q}$ splittings.

This leads to "sea quarks"

For partons with no valence contribution, we simply get



We can use the model to get quantitative predictions in the form of "sum rules"

Let $f_N(x) = u_p(x)$ for up quark content of proton, etc.

Ex: isospin $\Rightarrow \frac{1}{2}\int_0^1 dx \Big( (u_p(x) - \bar{u}_p(x)) - (d_p(x) - \bar{d}_p(x)) \Big) = \frac{1}{2}$

strangeness $\Rightarrow \int_0^1 dx \left( s_p(x) - \bar{s}_p(x) \right) = 0$

Charge: $\int_0^1 dx\, \frac{2}{3}\left(u_p(x) - \bar{u}_p(x)\right)$

$$- \int_0^1 dx\, \frac{1}{3}\left(d_p(x) - \bar{d}_p(x)\right) - \int_0^1 dx\, \frac{1}{3}\left(s_p(x) - \bar{s}_p(x)\right) = 1$$

etc. We can also develop momentum sum rules assuming only quarks. These do not agree with experiment at the $\mathcal{O}(50\%)$ level $\Rightarrow$ we need a non-trivial gluon content to describe the structure of nucleons! (See Cheng and Li; Sec 7.2)